$R_c \approx 0{,}5861\ l_c\ (L)$.

When $L > 1$, if $p < p_*(L)$, the stationary wave is stable irrespective of the shape of the cylinder. When $p > p_*(L)$, we determine the critical size of the square as $l_c = \pi/\sqrt{\mu^*(L, p)}$, where $\mu^*(L, p)$ is a transformation of the formula for the neutral hypersurface $p = p(\mu, L)$. For a specified L and $p > p_*(L)$, a stationary wave in a cylinder of square cross-section with a length of a side of the square $l < l_c$ is stable. When $l = l_c$ there is a loss of stability and, when there is a "short" perturbation, the solution corresponding to this instability is a stationary wave which is now inhomogeneous with respect to the variables x and y.

The author thanks G.G. Chernyi for discussing the results.

REFERENCES

1. GRISHIN A.M., BERTSUN V.N. and AGRANAT V.M., Investigation of the diffusion-thermal instability of laminar flames, Dokl. Akad. Nauk SSSR, 235, 3, 1977.
2. GRISHIN A.M., AGRANAT V.M. and BETSUN V.N., Diffusion-thermal instability of laminar flames in tubes, Dokl. Akad. Nauk SSSR, 241, 4, 1978.
3. MAKSIMOV YU.M., PAK A.T. LAVRENCHUK G.V. et al., Spin combustion of gas-free systems, Fizika Goreniya i Vzryva, 15, 3, 1979.
4. STRUNINA A.G. and DVORYANKIN A.V., The effect of thermal factors on the mechanism of the unstable combustion of gas-free systems, Dokl. Akad. Nauk SSSR, 260, 5, 1981.
5. SHKADINSKII K.G., KHAIKIN V.I. and MERZHANOV A.G., Propagation of the pulsed front of an exothermic reaction in a condensed phase, Fiz., Goreniya i Vzryva, 7, 1, 1971.
6. AVDEYEV P.A., Investigation of the stability of a stationary front of an exothermic reaction in a condensed phase, Izv. Akad. Nauk SSSR, MZhG, 1, 1985.
7. BALTES H.P. and HILF E.R., Spectra of Finite Systems. Bibliogr. Inst., Manheim, 1976.

Translated by E.L.S.

PMM U.S.S.R.,Vol.51,No.1,pp.20-28,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00
© 1988 Pergamon Press plc

ON THE THEORY OF THE FILTRATION OF A LIQUID IN A POROUS MEDIUM UNDER BULK HEATING BY A HIGH-FREQUENCY ELECTROMAGNETIC FIELD*

XUONG NGOC HAI, A.G. KUTUSHEV and R.I. NIGMATULIN

The process of the filtration and warming up of an extremely viscous liquid (bitumen) in a porous medium where there is a bulk thermal source due to the absorption of energy from a high-frequency electromagnetic field (hfemf) is investigated. This problem is associated with the analysis of bituminous oils /1/, the filtration of which is only realized in practice after a preliminary heating of the reservoir with the help of a hfemf, for example /2-5/.

It is assumed that the bitument is initially either in the liquid (mobile) or solid (immobile) state. Under the action of the bulk thermal source, the bitumen is heated, whereupon it melts, expands, flows, and moves with respect to the immobile, solid, porous skeleton of the rock under the pressure differential which is created. A closed system of differential equations is obtained and fundamental dimensionless similarity criteria are established which characterize the above-mentioned processes. The different types of stationary or limiting solutions which are realized during stationary or sufficiently lengthy heating of the medium are studied. When they exist, these solutions may be used to estimate the effectiveness of the actual process (to estimate the limiting length of the fusion zone, the extent of heating of the liquid bitumen and the characteristic time required for the process to attain a stationary state, etc., for example) and as tests to check the correctness of the various approximate and numerical methods for solving the resulting system of non-linear differential equations.

**Prikl.Matem.Mekhan.*,51,1,29-38,1987

The problem being considered differs from the classical Stefan problem on fusion and solidification processes /6, 7/ where it is assumed that the system is a single-component system, that there is no motion of the medium or convective transfer of the heat associated with it and that external supply of heat only takes place on the boundaries of the region under investigation (i.e. there is no bulk heat source present).

1. Basic assumptions and equations. The processes involved in the heating, fusion, and filtration of bitumen are investigated by the methods of the mechanics of multiphase continuous media under the following basic assumptions: the temperatures of the phases are the same in each elementary volume element of the porous medium; a phase transition (fusion or solidification) occurs on the discontinuity which separates the zones of the porous filled with the liquid and the solid bituminous phases (that is, there are no zones where the liquid and solid bituminous phases mix); the motion of the molten bitument in the rock is inertialess and obeys Darcy's law; there is no change in the volume of the non-molten (solid) phase or any deformation of the rock skeleton. The basic equations for the conservation of the masses and momenta of the phases and the energy of the mixture, subject to the above-mentioned assumptions, are considered in the case of a one-dimensional symmetric motion ($\nu = 0, 1,$ and 2 correspond to the cases of planar, cylindrical and spherical symmetry for the motion) in an Euler coordinate system /8-10/.

We shall employ the following notation. The subscripts $i = 1, 2,$ and 3 refer respectively to the parameters of the mobile liquid bitumen phase, the immobile (solid) bitumen phase and the rock skeleton. The subscript s refers to parameters on the melting line and the indices 0 and b refer to the parameters of the initial state and on the boundary of a pore respectively. We denote by ρ_i, u_i, c_i and λ_i the true density, the rate of filtration, the specific heat capacity and the thermal conductivity of the i-th phase ($i = 1, 2$ and 3). Further p and μ_1 are the pressure and viscosity of the liquid, k and m are the permeability and porosity of the rock ($k, m = \text{const}$), T is the temperature, x_b is the coordinate of the wall of the pore and $x_s(t)$ is the coordinate of the melting surface. We also introduce the quantities $(\rho c)^{(i)} = m\rho_i c_i + (1-m)\rho_3 c_3$, $\lambda^{(i)} = m\lambda_i + (1-m)\lambda_3$, $m = \text{const}$, $i = 1, 2$. In the region of the mobile phase, that is, the liquid (molten bitumen) phase, $x_b < x < x_s(t)$, $T > T_s$. We shall write the equations for the conservation of mass, momentum and energy in the form

$$\frac{\partial}{\partial t}(m\rho_1) + \frac{1}{x^\nu}\frac{\partial}{\partial x}(\rho_1 u_1 x^\nu) = 0, \quad u_1 = -\frac{k}{\mu_1}\frac{\partial p}{\partial x} \tag{1.1}$$
$$(\rho c)^{(1)}\frac{\partial T}{\partial t} + \rho_1 c_1 u_1 \frac{\partial T}{\partial x} = \frac{1}{x^\nu}\frac{\partial}{\partial x}\left(\lambda^{(1)} x^\nu \frac{\partial T}{\partial x}\right) + q^{(e)} + A^{(\tau)}$$

where $A^{(\tau)}$ is the work of viscous forces, $q^{(e)}$ is the intensity of the bulk thermal source due to the absorption of energy from a high-frequency electro-magnetic field. We next consider the case when $A^{(\tau)} \ll q^{(e)}$ and the quantity $A^{(\tau)}$ may be neglected. The viscosity of the liquid is temperature dependent ($\mu_1 = \mu_1(T)$).

In the immobile (solid bitumen) phase region $x > x_s(t)$, $t > t_s$, we have

$$(\rho c)^{(2)}\frac{\partial T}{\partial t} = \frac{1}{x^\nu}\frac{\partial}{\partial x}\left(\lambda^{(2)} x^\nu \frac{\partial T}{\partial x}\right) + q^{(e)}, \quad u_2 = 0, \quad \rho_2 = \text{const} \tag{1.2}$$

On the boundaries of the region under investigation and the melting front

$$x = x_b, \quad p = p(x_b, t), \quad \lambda^{(1)} S_b \frac{\partial T}{\partial x}\Big|_{x = x_b + 0} = -q_b \tag{1.3}$$
$$x = x_s(t), \quad T = T_s = \text{const}$$
$$\frac{k}{\mu_1}\frac{\partial p}{\partial x}_{x = x_s - 0} = -u_{1s} = -m\left(1 - \frac{\rho_2}{\rho_{1s}}\right)\frac{dx_s}{dt};$$
$$x \to +\infty, \quad T \to T_0$$

where $q_b = q(x_b, t)$ is the intensity of the total thermal flux through the boundary x_b ($q_b > 0$ corresponds to the case of the input of heat and $q_b < 0$ to the case when heat is extracted).

In the case when $T_0 < T_s$ and $t < t_s$, the temperature distribution within the medium in the region $[x_b, +\infty) \times [0, t_s]$ is found from the solution of Eqs.(1.2) with the boundary conditions

$$x = x_b, \quad \lambda^{(2)} S_b \frac{\partial T}{\partial x}\Big|_{x = x_b + 0} = -q_b; \quad x \to +\infty, \quad T \to T_0 \tag{1.4}$$

The distribution of thermal sources $q^{(e}$ which arise due to the absorption of energy from the high frequency electromagnetic field is determined from the Poynting equation and the Bouger-Lambert law for a monochromatic wave

$$q^{(e)} = -\nabla\cdot\mathbf{R}, \quad \nabla\cdot\mathbf{R} = \mathbf{R}/L_R \tag{1.5}$$

where $\mathbf{R}$ is the radiation intensity vector and L_R is the absorption length, which characterizes the degree of absorption of energy from the electromagnetic wave by the medium under consideration. In the case of the propagation of a one-dimensional (planar, cylindrical or spherical) monochromatic wave in a homogeneous and isotropic medium, these equations have the form

$$q^{(e)} = -\frac{1}{x^{\nu}}\frac{\partial}{\partial x}(x^{\nu}R) = \frac{R}{L_R} \quad (1.6)$$

In the general case the absorption length L_R for a specified medium is determined by the frequency of the electromagnetic radiation ω and depends on the pressure p and the temperature. Then, even at a fixed frequency ω, the equations for the thermohydrodynamic parameters (1.1) and (1.2) and Eq. (1.6) for $q^{(e)}$ and the electrophysical parameter R are interrelated and must be solved jointly. The effect of pressure and temperature on the absorption length L_R can frequently be neglected. The quantity L_R, at a fixed frequency ω, then becomes a parameter which is known in advance and immediately determines the intensity of the radiation R and the intensity of the bulk thermal source independently of the solution of the thermohydrodynamic equations. Here

$$q^{(e)} = \frac{R_b}{L_R}\left(\frac{x_b}{x}\right)^{\nu}\exp\left(-\frac{x-x_b}{L_R}\right) \quad (1.7)$$

R_b is the intensity of the radiation on the pore boundary x_b which is determined by the power $N^{(e)}$ and surface area of the radiator S_b (for $\nu = 0$ $S_b = 1$ m^2, for $\nu = 1$ $S_b = 2\pi x_b h$ (h is the power of the radial reservoir) and, for $\nu = 2$ $S_b = 4\pi x_b^2$).

The position of the melting surface is determined from the conditions of mass balance and in the quasistatic (filtration) approximation of the energy balance on the interphase boundary

$$dx_s/dt = j/m\rho_2, \quad jl = q_{1s} + q_{2s} \quad (1.8)$$
$$q_{1s} = -\lambda^{(1)}\frac{\partial T}{\partial x}\Big|_{x=x_s(t)-0}, \quad q_{2s} = \lambda^{(2)}\frac{\partial T}{\partial x}\Big|_{x=x_s(t)+0}$$

Here q_{1s} and q_{2s} are the thermal fluxes arriving at the interphase surface from the side of the mobile and immobile phases, and j and l are the intensity and specific heat of the phase transition (the difference in the enthalpies of the phases) respectively.

Linear equations of state are assumed for the density of the liquid

$$\rho_1(p, T) = \rho_{10} + \beta_p(p - p_0) - \beta_T(T - T_0) \quad (1.9)$$

where β_p, β_T are the coefficients of compressibility and thermal expansion. Then, the system of thermohydrodynamic Eqs. (1.1)-(1.4), (1.7)-(1.9), subject to the initial conditions

$$t = 0, \quad p = p_0, \quad T = T_0 \quad (1.10)$$

is closed and can be solved.

In the general case the liquid parameters on the fusion front must be determined using the Clausius-Clapeyron equation from the condition that the phases are in equilibrium. In the majority of important practical cases, when the melting pressure does not change to any great extent: $p - p_0 \ll p^\circ = \rho_{1s}\rho_2(\rho_2 - \rho_{1s})^{-1} l$, the condition $T_s = \text{const}$ can be used and this has been done in the present paper.

2. Dimensionless variables and parameters. The following dimensionless variables and parameters are introduced:

$$\tau = \frac{u_* t}{L_R}, \quad X = \frac{x}{L_R}, \quad X_s(\tau) = \frac{x_s(t)}{L_R}, \quad U_i = \frac{u_i}{u_*}, \quad \Phi_i = \frac{\rho_i}{\rho_*}$$
$$\Theta = \frac{T}{T_s}, \quad P = \frac{p}{p_*}, \quad M_1(\Theta) = \frac{\mu_1(T)}{\mu_1(T_s)}, \quad u_* = \frac{k}{\mu_1(T_s)}\frac{p_*}{L_R}$$

The subscript * refers to certain characteristic parameters of the medium.

In the new variables the system of Eqs. (1.1)÷(1.4), (1.7)-(1.9) is transformed to two differential equations of parabolic type in the region $X_b < X < X_s(\tau)$ and a single equation of a similar type in the region $X > X_s(\tau)$, the solution of which, when the dissipation of energy due to the viscosity of the liquid is neglected ($A^{(\nu)} = 0$), depend on the following dimensionless parameters which, respectively, characterize:

the geometrical properties of the space m, ν, X_b

the thermophysical properties of the phases:

$$B_p = \frac{\beta_p p_*}{\rho_*}, \quad B_T = \frac{\beta_T T_s}{\rho_*}, \quad C_2 = \frac{c_2}{c_1}, \quad C_3 = \frac{\rho_3 c_3}{\rho_* c_1},$$
$$\delta = \frac{\rho_2 - \rho_{1s}}{\rho_{1s}}$$

the relative effect of convective energy transfers compared with energy transfer due to heat conduction:

$$\mathrm{Pe}_i = u_* L_R / D_i \quad (D_i = \lambda^{(i)}/(\rho_* c_1), \quad i = 1, 2)$$

(Pe_i and D_i are the Péclet number and the thermal conductivity of the i-th phase);

the relative contribution to the change in the energy of the medium from heat conduction and fusion processes;

$$G_1 = u_* L_R / \Lambda_i \quad (\Lambda_i = \lambda^{(i)} T_s / (\rho_* l), \quad i = 1, 2)$$

the effect of the external input of heat and the boundary conditions:

$$N = \frac{N^{(e)}}{S_b \rho_* c_1 u_* T_s}, \quad \Theta_0, \quad Q_b = \frac{q_b L_R}{\lambda^{(1)} S_b T_s}.$$

Assuming that the liquid pahse is incompressible, the problem under consideration reduces to a problem on the melting of a substance taking into account convective thermal conductivity in the liquid and the existence of an external bulk heat source. In this case the velocity and pressure fields in the liquid phase are expressed in terms of the temperature field in the following manner: $X_b < X < X_s$

$$U_1 = -m\delta \left(\frac{X_s}{X}\right)^\nu \frac{dX_s}{d\tau}, \quad P = 1 + m\delta X_s^\nu \frac{dX_s}{d\tau} \int_{X_b}^{X} \frac{M_1(\Theta)}{\xi^\nu} d\xi$$

When the change in the viscosity of the liquid phase with temperature is neglected $(M_1(\Theta) = 1)$, the pressure field in the liquid is found in an elementary manner. For example, when $\nu = 1$,

$$P = 1 + m\delta X_s \frac{dX_s}{d\tau} \ln \frac{X}{X_b}, \quad X \in [X_b, X_s).$$

3. Stationary solutions with fixed melting and solidification fronts. Let us first consider stationary or limiting solutions of the type

$$\partial\rho_i/\partial t = 0, \quad \partial T/\partial t = 0, \quad x_s = \text{const} \quad (i = 1, 2) \tag{3.1}$$

which correspond to the case of the continuous and constant heating of a porous medium which is initially saturated with solid bitumen. In this case $u_1 = 0$ and the problem reduces to solving the system of ordinary differential equations with the boundary conditions

$$\frac{d}{dX}\left(X^\nu \frac{d\Theta}{dX}\right) = -K_i \exp[-(X - X_b)] \quad (i = 1, 2) \tag{3.2}$$

$$\begin{gathered} K_i = \mathrm{Pe}_i N X_b^\nu \\ X \in [X_b, X_s), \quad U_1 = 0, \quad P = 1, \quad \Phi_1 = 1 - B_T(\Theta - \Theta_0) \\ X \in [X_s, +\infty), \quad U_2 = 0, \quad \Phi_2 = \text{const} \\ X = X_b, \quad (d\Theta/dX)_{X = X_b + 0} = -Q_b = \text{const} \\ X = X_s, \quad \Theta = 1, \quad j = 0, \quad \frac{1}{G_1}\frac{d\Theta}{dX}\Big|_{X = X_s - 0} = \frac{1}{G_2}\frac{d\Theta}{dX}\Big|_{X = X_s + 0} \\ X \to +\infty, \quad \Theta \to \Theta_0 \quad (\Theta_0 < 1) \end{gathered} \tag{3.3}$$

The general solution of Eq. (3.2) has the form

$$\begin{gathered} \nu = 0, \quad \Theta(X) = -K_i \exp[-(X - X_b)] + CX + D \\ \nu = 1, \quad \Theta(X) = K_i \exp X_b \,\mathrm{Ei}(-X) + C \ln X + D \\ \nu = 2, \quad \Theta(X) = -K_i \exp X_b F(X) - C/X + D \\ F(X) = \exp(-X)/X + \mathrm{Ei}(-X) \\ \mathrm{Ei}(-X) = -\int_X^{+\infty} \frac{d\xi}{\xi e^\xi} < 0, \quad X > 0 \end{gathered} \tag{3.4}$$

where K_i is a dimensionless parameter which characterizes the ratio of the heat evolution and heat conduction, C and D are constants of integration, and $\mathrm{Ei}(-X)$ is the integral exponential function. We note that, usually, $X_b \ll 1$ and therefore $\exp X_b \approx 1$.

Assertion 1. No solution of the system of Eqs. (3.2) which satisfies conditions (3.3) exists for $\nu = 0, 1$.

Proof. It is obvious from the general solution (3.4) that, in order to satisfy the last condition of (3.3), it is necessary to put $C = 0, D = \Theta_0$. By virtue of the inequalities $K_i \exp[-(X - X_b)] < 0$ (for $\nu = 0$) and $K_i \exp X_b \,\mathrm{Ei}(-X) < 0$ (for $\nu = 1$) when $X \in [X_b, +\infty)$, it follows that, for any $X \in [X_b, +\infty)$, we have $\Theta(X) \leqslant \Theta_0$, that is, no coordinate X_s exists such that $\Theta(X_s) = 1 > \Theta_0$ when $X = X_s$.

Assertion 2. A unique solution of the system of Eqs. (3.2) which satisfies conditions (3.3) exists for $\nu = 2$ when $K_2 \geqslant K_* = -(1-\Theta_0)/[\exp(X_b)\cdot \mathrm{Ei}(-X_b)]$ and $Q_b = 0$. At the same time $X_s > X_b$ in the case when $K_2 > K_*$.

Proof. For $\nu = 2$, the temperature profile in the medium (3.4) satisfies conditions (3.3) if the constants of integration are defined in the following manner:

$$X \in [X_b, X_s), \quad C = -(K_1 + Q_b X_b^2), \quad D = 1 + C X_s^{-1} + K_1 \exp X_b \, F(X_s) \tag{3.5}$$

$$X \in [X_s, +\infty), \quad C = -X_s[1 - \Theta_0 + K_2 \exp X_b F(X_s)], \quad D = \Theta_0 \tag{3.6}$$

and the equation

$$Z(X) = K_1^{-1} Q_b X_b^2 + V(X) - XY(X) = 0 \tag{3.7}$$

$$V(X) = 1 - \exp[-(X - X_b)] \quad Y(X) = K_2^{-1}(1 - \Theta_0) - \exp X_b \, \mathrm{Ei}(-X)$$

is obtained from (3.3) for determining the site of the melting front X_s. (In this case, the terms containing Q_b are equal to zero. Relationships (3.5)-(3.7) are also true in the case when $Q_b \neq 0$.)

Let us prove the existence and uniqueness of the solution of Eq. (3.7) when $Q_b = 0$. We consider the functions $V(X)$ and $XY(X)$. When $K_2 \geqslant K_*$, it follows, by virtue of the monotonicity of the negative increasing function $\mathrm{Ei}(-X)$, that the equation $Y(X) = 0$ has a unique root $X = X_*$ in the region $X > X_b$ and, moreover, if $K_2 = K_*$, then $X_* \equiv X_b$. In the region $X \in [X_*, +\infty)$ $XY(X)$ is a non-negative function which increases monotonicity from 0 to $+\infty$ and $V(X)$ is a bounded, positive, increasing function. In the above-mentioned region, $d(XY)/dX > dV/dX$, whence it follows that, in this region, the graphs of the functions $XY(X)$ and $V(X)$ intersect at one and only one point which corresponds to the coordinate of the melting front X_s and where $V(X) - XY(X) = 0$. In the region $X \in [X_b, X_*)$, we have $XY(X) < 0$, $V(X) \geqslant 0$, i.e. the graphs of these functions do not intersect one another (the right-hand upper part of Fig.1).

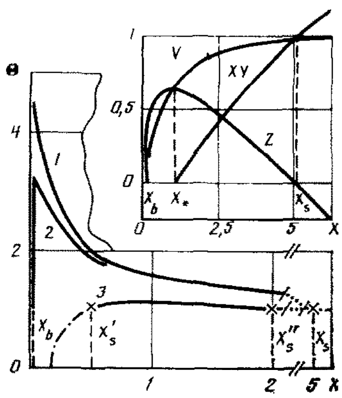

Fig.1

If $Q_b \neq 0$, an analogous assertion can be proved in the following cases:

1) $K_2 < K_*$, $Q_b > 0$ and $Q_b X_b^2 - K_1 X_b Y(X_b) \geqslant 0$;

2) $K_2 \geqslant K_*$, $Q_b \geqslant 0$ (and assertion 2 corresponds to the special case when $Q_b = 0$);

3) $K_2 \geqslant K_*$, $Q_b < 0$ and $|Q_b X_b^2| < K_1 X_b Y(X_b)$

In the case when $K_2 \geqslant K_*$, $Q_b < 0$, $K_1 X_b Y(X_b) \leqslant |Q_b X_b^2| < K_1 V(X_*)$, the existence and uniqueness of a two-profile solution of problem (3.2) with the following conditions:

$$\begin{aligned}
&X \in [X_b, X_s'), \quad U_2 = 0, \quad \Phi_2 = \mathrm{const} \\
&X \in [X_s', X_s''), \quad U_1 = 0, \quad P = 1, \quad \Phi_1 = 1 - B_T(\Theta - \Theta_0) \\
&X \in [X_s'', +\infty), \quad U_2 = 0, \quad \Phi_2 = \mathrm{const} \\
&X = X_b, \quad (d\Theta/dX)_{X = X_b + 0} = -Q_b = \mathrm{const} \\
&X = X_s', \quad \Theta = 1, \quad j = 0, \quad \frac{1}{G_2}\frac{d\Theta}{dX}\Big|_{X = X_s' - 0} = \frac{1}{G_1}\frac{d\Theta}{dX}\Big|_{X = X_s' + 0} \\
&X = X_s'', \quad \Theta = 1, \quad j = 0, \quad \frac{1}{G_1}\frac{d\Theta}{dX}\Big|_{X = X_s'' - 0} = \frac{1}{G_2}\frac{d\Theta}{dX}\Big|_{X = X_s'' + 0} \\
&X \to +\infty, \quad \Theta \to \Theta_0 \quad (\Theta_0 < 1)
\end{aligned} \tag{3.8}$$

can be proved.

In this case the coordinates of the fronts X_s', X_s'' $(X_s' < X_s'')$ are determined by the same Eq. (3.7) which now has two roots and the constants of integration take the following values: when $X \in [X_b, X_s')$ they are the same as in (3.5) with the replacement of K_1 by K_2 and X_s by X_s', when $X \in [X_s'', +\infty)$, they are the same as in (3.6) with the resplacement of X_s by X_s'' and, when $X \in [X_s', X_s'')$:

$$C = -K_1 \exp X_b \frac{X_s' \cdot X_s''}{X_s'' - X_s'}[F(X_s') - F(X_s'')]$$

$$D = 1 - K_1 \exp X_b \frac{1}{X_s'' - X_s'}[X_s' F(X_s') - X_s'' F(X_s'')]$$

No stationary solution of the given type exists in the remaining cases. We note that the parameter K_* characterizes, when the rest of the conditions are identical, the possibility of the establishment of a stationary (immobile) melting surface in a medium which is subjected to long-term heating by thermal sources of the type of (1.7) with powers $K_2 \geqslant K_*$.

Several examples of the solution of problems (3.2)-(3.3) and (3.2), (3.8) are presented in Fig.1 for the vales $K_1 = 0.82$, $K_2 = 0.75$ which correspond, for example, to the following values of the phase parameters and the parameters of the spatial heat sources: $T_0 = 293$ K, $T_s = 343$ K, $\lambda_1 = 0.6$ W/m.K, $\lambda_2 = 1.2$ W/m.K, $\lambda_3 = 5.8$ W/m.K, $m = 0.5$, $L_R = 25$ m, $N^{(e)} = 280$ kV, $x_b = 0.15$ m.

Curve *1* corresponds to the single profile solution of problem (3.2) when there is no input of heat into the reservoir or withdrawal of heat from it through the boundary of the region $X = X_b: Q_b = 0$, which is depicted in the right-hand upper part of Fig.1. Curve *2* illustrates the limiting single profile solution in the case when heat is extracted on the boundary $X = X_b$: $Q_b = -K_1X_b^{-1}Y(X_b) \approx -0.026\, K_1X_b^{-2}$. Curve *3* demonstrates the two-profile solution which occurs when a more significant amount of heat is extracted through the boundary $X = X_b$: $Q_b \approx -0.573\, K_1X_b^{-2}$.

It can be seen (curve *1*) that the maximum region in which the medium (bitumen) experience a phase transition as the result of the constant action of a spatial heat source (a high-frequency electromagnetic field) with a power of 280 kV. covers a considerable distance of about 5.1 L_k, that is, around 125 m. The temperature of the liquid bitumen in the neighbourhood of a pore attains values of $\approx 4.5\, T_s$, that is, ≈ 1500 K. In the limiting case of the single profile problem, when the greatest extraction of heat through the left-hand boundary of the investigated region is realized, the temperature of the bitumen on the wall of the pore is lowered to the melting point $X = X_b$: $T_b = T_s = 343$ K (curve *2*). At the same time a sharp lowering of the temperature of the bitumen is observed in a very narrow zone close to the interstice. Thus, the peak value of the temperature of the liquid bitumen, which is attained at a distance of about 0.75 m, is reduced to 3.2 T_s, that is, down to ≈ 1100 K but the melting front is only displaced towards the centre insignificantly (by an account $\approx 0.1\, L_R$, that is, ≈ 2.5 m).

A further increase in the amount of heat Q_b which is extracted through the boundary $X = X_b$ leads to the appearance of a solidification front (curve *3*) but, at the same time, negative temperatures are realized close to the interstice. The actual maximum discharge of heat from the productive reservoir through the boundary $X = X_b$ due to the thermal conductivity of the medium is attained when $\Theta(X_b) = 0$. In this case the temperature profile in the medium is practically identical to curve *2* which suggests that the real possibility of the discharge of heat on the boundary of an interstice only has a small effect on the process.

The parameters K_i $(i = 1, 2)$ are the basic dimensionless criteria in stationary solutions of the type of (3.1). In the case when $j = 0$, the dimensionless parameters G_1 and G_2 which occur in conditions (3.3) or (3.8) are expressed in terms of them:

$$G_1/G_2 = \mathrm{Pe}_1/\mathrm{Pe}_2 = K_1/K_2.$$

The quantity L_R occurring in the parameters K_i may vary due to a change in the frequency ω of the electromagnetic radiation. When L_R is increased, there is an increase in the limiting melting zone but, simultaneously, there is an increase in the time of its establishment. Other conditions being the same, the maximum practicable melting zone is attained under the action of a high-frequency electromagnetic field with a frequency which corresponds to the maximum length of the relaxation zone L_R.

It should be noted that, in cases when a stationary solution exists, it is possible to estimate the characteristic time of emergence onto a stationary regime, t_{st}. For example, in the spherical case ($\nu = 2$) when the medium is solely heated by a high-frequency electromagnetic field ($q_b = 0$), the following estimate

$$t_{st} \sim Q_{st}/N_{st}$$

$$Q_{st} \approx \Omega_s \{(\rho c)^{(1)} (\langle T\rangle - T_s) + m\rho_2 l + (\rho c)^{(2)} (T_s - T_0)\}$$

$$N_{st} \approx N^{(e)} \left[1 - \exp\left(-\frac{x_s - x_b}{L_R}\right)\right]; \quad \Omega_s = \frac{4}{3}\pi (x_s^3 - x_s^3)$$

holds on the basis of the single front solution (3.4)-(3.7).

Here N_{st} and Q_{st} are the intensity and the overall amount of heat supplied into the limiting stationary zone of the molten bitumen with a volume Ω_s and a mean temperature $\langle T\rangle$.

In the case when a productive reservoir is heated by inputting energy through its boundary $X = X_b$ solely due to the thermal conductivity of the medium ($N = 0$, $Q_b > 0$), the coordinate of the stationary melting front of the bitumen is determined in the following manner:

$$X_s^{(q)} = \frac{Q_b X_b^2}{1 - \Theta_0}, \quad Q_b > Q_* = \frac{1 - \Theta_0}{X_b^2} \tag{3.9}$$

On the basis of (3.9), it may be shown that, when one and the same amount of energy is fed into reservoirs, the remaining parameters being the same, the limiting depth of penetration of the melting front, in the case of the input of heat through the boundary $X = X_b$ which is solely due to the thermal conductivity of the medium $X_s^{(q)}$ $(Q_b > 0,\ N = 0)$, will be greater than the corresponding value in the case when the reservoir is heated solely by a spatial heat source of the type of (1.7) X_s $(Q_b = 0,\ N \neq 0)$. At the same time, the temperature around the interstice in the first case $T^{(q)}(X_b)$ is far greater than that in the second case $T(X_b)$.

In the example which has been illustrated by curve *1* in Fig.1, $X_s \approx 5.1\ L_R$, $T(X_b) \approx 4.5\ T_s$ while $X_s^{(q)} \approx 5.14\ L_R$, $T^{(q)}(X_b) \approx 136$ and $T_s \approx 30\ T(X_b)$. The insignificant difference which exists in the solutions X_s and $X_s^{(q)}$ is due to the fact that, when the reservoir is heated by means of a spatial heat source, a large amount of the radiative energy is transmitted into the region where the bitumen is solid. In this case, however, a temparture distribution of the medium throughout the space is obtained which is more uniform and the time required for the establishment of the stationary solution is, obviously, much shorter.

The stationary solutions which have been obtained are characterized by the existence of a singularity at the centre of the region investigated: $\Theta(X_b) \to +\infty$ as $X_b \to 0$. The existence of such a singularity may manifest itself in a tangible manner in the fact that a very pronounced increase in the temperature of the medium in the neighbourhood of an interstice will be observed as the size of the high-frequency electromagnetic field energy generator is reduced. In this case, as in the case represented in Fig.1, on account of the high temperature of the medium around the interstice the solutions which have been given may be found to be unacceptable due to the need to take account of such additional processes as the deformation of the rock skeleton, gas formation, cracking, etc.. On the basis of the general solution (3.4), the existence and uniqueness of the solutions of analogous problems in a finite region of space $[X_b, X_k]$, when different laws are specified regarding the input and extraction of heat through the boundaries $X = X_b$ and $X = X_k$, may also be demonstrated.

It should be noted that, in the general case, it is necessary to take account of the reflection of the electromagnetic wave from the melting (solidification) front and the difference in the absorption coefficients of the waves for the solid and liquid bitumens. Allowance for these factors on the basis of the system of Eqs.(1.1)-(1.4), (1.7)-(1.10) does not create any great difficulties and does not have any fundamental effect on the basic results which have been obtained.

4. Stationary solutions of problems of the filtration of a viscous liquid in the field of a spatial heat source. Let us consider stationary solutions of the type

$$\partial\rho_1/\partial t = 0,\quad \partial T/\partial t = 0,\quad u = u_1(x)\quad (x_s \to +\infty) \tag{4.1}$$

which correspond to the case when there is no solid phase $(\mu_2 < +\infty)$ or surface of melting in general. The resulting system of equations then takes the following dimensionless form:

$$-G_b\frac{d\Theta}{dX} = \frac{1}{\mathrm{Pe}_1}\frac{d}{dX}\left(X^\nu\frac{d\Theta}{dX}\right) + NX_b{}^\nu\exp[-(X - X_b)] \tag{4.2}$$

$$\frac{dP}{dX} = -M_1U,\quad U = -\frac{G_b}{\Phi_1X^\nu},\quad G_b = -(\Phi_1UX^\nu)|_{X=X_b} = \text{const} > 0$$

$$\Phi_1 = 1 + B_p(P-1) - B_T(\Theta - 1)$$

$$X = X_b,\quad (d\Theta/dX)_{X=X_b+0} = -Q_b \tag{4.3}$$

$$X \to +\infty,\quad \Theta \to \Theta_\infty < +\infty,\quad P \to P_\infty < +\infty$$

$$(\Theta = T/T_*)$$

In the case when $\nu = 0$ or 1, no solution of the system of Eqs.(4.2), exists which satisfies conditions (4.3). In fact, in this case, according to the second equation, the pressure increases without limit:

$$\nu = 0,\quad P^2 \sim \text{const}\ G_bX \to +\infty \tag{4.4}$$

$$\nu = 1,\quad P^2 \sim \text{const}\ G_b \ln X \to +\infty$$

which contradicts the last condition of (4.3).

It can be shown, however, that a solution of the system of equations (4.2) with the boundary conditions

$$X = X_b,\quad P = P_b,\quad (d\Theta/dX)_{X=X_b+0} = -Q_b;\quad X \to +\infty,\qquad \Theta \to \Theta_\infty < +\infty \tag{4.5}$$

exists which does not require any constraint on its asymptotic behaviour with respect to pressure as $X \to \infty$. At the same time, the temperature distributions of the liquid have the form

$$\nu = 0,\ \Theta(X) = 1 - \frac{K_1\exp[-(X - X_b)]}{1 - \varkappa}\times\left\{1 - \frac{\exp[-(X - X_b)\cdot(\varkappa - 1)]}{\varkappa}\right\} + Q_b\exp[-\varkappa(X - X_b)]/\varkappa$$

$$\nu=1,\quad \Theta(X)=1+\varkappa^{-1}K_1\left(\frac{X_b}{X}\right)^{\varkappa}\times\left[1+\frac{Q_bX_b}{K_1}+\varkappa\exp X_b\int_1^{X/X_b}\exp(-\xi X_b)\,\xi^{\varkappa-1}d\xi\right]$$

$$\varkappa=\mathrm{Pe}_1\,G_b$$

The existence of a solution of the system of Eqs.(4.2) with the asymptotic behaviour (4.4) as $X\to+\infty$ shows that, in practice when conditions (4.3) are satisfied, the stationary heating of a reservoir by a spatial energy source of the high-frequency electromagnetic field type (1.4) in the case when $\nu=0$ or 1 must be accompanied by a decrease in the delivery time of the liquid. In the limiting case when $G_b=0$, the system of Eqs.(4.2) reduces to a single heat conduction equation, the solution of which is readily obtained and which uniquely relates the density distribution of the liquid to its temperature distribution. At the same time the pressure in the liquid will be homogeneous; $P(X)=\mathrm{const}$.

In the case when $\nu=2$ it may be shown that a solution of problem (4.2) with the boundary conditions (4.3) exists and the temperature field is determined in the following manner:

$$\Theta(X)=1+\varkappa^{-1}K_1\exp X_b\{\exp(-X)+\exp(-X_b)[I(X)\cdot\exp(\varkappa/X)-I(+\infty)]\}+\varkappa^{-1}Q_bX_b^2(\exp(\varkappa/X)-1)\qquad(4.6)$$

$$I(X)=\int_{X_b}^{X}\exp[-(\xi+\varkappa/\xi-X_b)]\,d\xi$$

while the pressure field in the liquid can be found from the solution of the following ordinary differential equation:

$$G_b\frac{M_1}{X^2}\frac{dX}{dP}=1+B_p(P-1)-B_T(\Theta-1)\qquad(4.7)$$

$$X\to+\infty,\quad P\to P_\infty<+\infty$$

using, for example, a numerical method. In this case on account of the geometry of the space, a constant delivery of liquid can be achieved by means of the stationary heating of the reservoir by a spatial heat source of type (1.4).

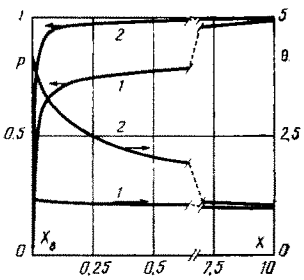

Fig.2

Fig.2 shows the solutions of problem (4.6), (4.7) for $K_1=0.71$: $\mathrm{Pe}_1=2.97$, $B_p=0.002$; $B_T=0.15$; $X_b=0.006$ and $Q_b=0$ which correspond, for example, to: $p_b=0.1$ MPa, $p_*=p_\infty=3.0$ MPa, $T_*=T_\infty=300$ K, $\rho_*=\rho_\infty=10^3$ kg/m^3, $\mu_*=1.0$ Pa.s, $c_1=2.1$ kj/kg.K, $\lambda_1=0.60$ W/m.K, $\lambda_3=5.8$ W/m.K, $k=2.0$ μm^2, $m=0.3$; $N^{(e)}=280$ kV, $q_b=0$; $L_R=25$ m and $x_b=0.15$ m.

Curves *1* and *2* illustrate the temperature and pressure distributions in the medium for the cases when $\mu_1=(T/T_\infty)^{-18}$ which is characteristic in the case of a bituminous oil (curve *1*) and when $\mu_1=\mu_\infty=\mathrm{const}$ (curve *2*) while the deliveries G_b are equal to 0.58 ($\approx$40ton/day) and 0.003 (≈0.2 ton/day). It can be seen that, when account is taken of the effect of temperature on the viscosity of the liquid, this leads to a substantial change in the pressure and temperature fields. In this case the temperature of the medium around an interstice, on account of the more intense heat transfer, is reduced several-fold compared with the case when $\mu_1=\mu_\infty=\mathrm{const}$. At the same time, the characteristic length of the active filtration zone is of the order of $10L_R$, i.e. ≈250 m.

In conclusion we note that, although information on the stationary solution is useful for estimating the efficiency of the process the study of the non-stationary stage and the determination of the dynamics of the process, in particular, when the intensities or the action vary with time and finding the optimal laws of irradiation as a function of the properties of the medium are of greater interest and more urgent from a theoretical and practical point of view.

REFERENCES

1. VAKHITOV G.G. and SAFIULLIN R.KH., The Opening up of Bitumen Deposits, Summaries of Science and Technology. Series Development of Petroleum and Gas Deposits (Itogi nauki i tekhniki, Ser, Razrabotka neftyanykh i gazovykh mestorozhdenii), 10, Vses. Inst. Nauchn. i Tekhn. Informatsii (VINITI), Moscow, 1978.
2. SAYAKHOV F.L., FATYKHOV M.A. and KUZNETSOV O.L., Investigation of electromagnetic-acoustic action on the temperature distribution in petroleum-water saturated rock, Izv. Vuz Neft' i Gaz, 3, 1981.
3. CHISTYAKOV S.I., DENISOVA N.F. and SAYAKHOV F.L., Experimental investigation of the dependence of the dielectric properties of petroleum and its fractions on frequency, Izv. Vuz Neft' i Gaz, 5, 1972.
4. ALBERNETHY E.R., Production increase of heavy oil by electromagnetic heating, J. Canad. Petr. Tech., 15, 3, 1976.

5. BRIDGES J., STRESTY G., TAFLO' A. and SNOW R., Radio-frequency heating to recover oil from Utah tar-sands, in: Th 'uture of Heavy Crude Oils and Tar Sands, McGraw-Hill Inc., New York, 1980.
6. RUBINSHTEIN L.I., Stefan Prob[illegible]s (Problemy Stefana), Zvaigzne, Riga, 1967.
7. CARSLOW H.S. and JAEGER J.C., Conduction of Heat in Solids, Clarendon Press, Oxford, 1959, /Russian translation/, Nauka, Moscow, 1964.
8. NIGMATULLIN R.I., Fundamentals of the Mechanics of Heterogeneous Media, (Osnovy mekhaniki geterogennykh sred), Nauka, Moscow, 1978.
9. POLUBARINOVA-KOCHINA P.YA., Theory of the Motion of Ground Waters (Teoriya dvizheniya gruntovykh vod), Nauka, Moscow, 1977.
10. BARENBLATT G.I., ENTOV V.M. and RYZHIK V.M., The Motion of Liquids and Gases in Natural Reservoirs (Dvizheni zhidkostei i gazov v prirodnykh plastakh), Nedra, Moscow, 1984.

Translated by E.L.S.

PMM U.S.S.R., Vol.51, No.1, pp.28-33, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00
©1988 Pergamon Press plc

ISOLATION OF SINGULARITIES IN THE SOLUTION OF TWO-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY IN IRREGULAR MULTIPLY CONNECTED DOMAINS*

A.M. LEVIN

A general method is considered for isolating the singularities of the solutions of a plane problem of the theory of elasticity, a problem of the bending of thin elastic plates, and harmonic problems of the theory of elasticity in multiply connected domains with boundary breaks. The procedure is used to solve the problems by the method of compensating loads (MCL) or the method of integral equations of the first kind /1-6/.

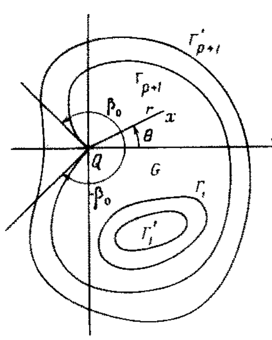

Fig.1

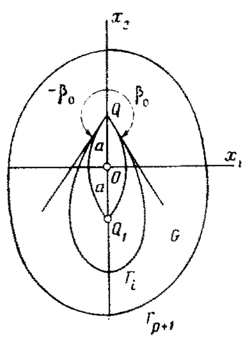

Fig.2

In the MCL the components of the directionally deformed state (DDS) are sought as potentials which are distributed along contours, spaced a certain distance from the domain boundary, rather than distributed along the boundary itself. When the potentials are substituted into the boundary conditions, systems of integral equations of the 1st kind are obtained in the unknown densities. Methods of regularizing the solution of these equations were considered in /3-6/. When the components of the DDS have singularities at corner points of the boundary, the modification of the MCL consists in adding to the potentials of the singular solutions of homogeneous boundary value problems for the auxiliary wedge-shaped domains /7-12/ (Fig.1). However, if the initial domain is multiply connected and the corner points are located on "interior" pieces of the boundary (Fig.2), the solutions for the wedge cannot be used in MCL. In this case the cut needed to isolate the one-valued branches of the

*Prikl.Matem.Mekhan.,51,1,39-46,1987